Modified Anderson-Darling detector for Spectrum Sensing

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Goodness-of-Fit (GoF) testing is a promising approach for blind spectrum sensing in cognitive radio (CR) applications. In this paper, we present a modification on the Anderson-Darling detector by taking into account the physical characteristics of spectrum sensing, resulting in a reduction of the false alarm rate by a factor of two for a given detection threshold in the binary hypothesis test.

Introduction: A GoF test is a blind non parametric hypothesis test which can be used to detect the presence of signals in noise by determining whether the received samples are (are not) drawn from a distribution with a Cumulative Distribution Function (CDF) F_0, representing the noise CDF [1, 2, 3]. The hypothesis to test whether or not a signal is present can be formulated as follows:

$$\begin{cases} H_0 : F_n(x) = F_0(x) \\ H_1 : F_n(x) \neq F_0(x), \end{cases} \quad (1)$$

where F_0 represents the hypothesized noise CDF. $F_n(x)$ is the empirical CDF of the received sample and can be calculated by:

$$F_n(x) = |\{i : x_{(i)} \leq x, 1 \leq i \leq n\}/n|, \quad (2)$$

where $|.|$ indicates cardinality, $x_{(i)}$ are the ordered samples under test ($x_1 \leq x_2 \leq .... \leq x_n$) and n represents the total number of samples.

In statistics, the aim of GoF is to test the hypothesis that samples have been drawn from a population with a specified continuous CDF $F_0(x)$. This means that in the H_1 hypothesis, for a given value of x, the discrepancy between $F_n(x)$ and the hypothetical $F_0(x)$ can either be positive or negative (the empirical CDF $F_n(x)$ can be left or right from $F_0(x)$ in a graphical representation). However, in the application of spectrum sensing this is not necessarily true, as will be shown in this paper.

The remainder of this paper is organized as follows. In the next section, the Anderson-Darling (AD) test statistic is detailed, along with a description of how the AD detector can be used for spectrum sensing. In the following section, the shortcoming of the AD test for spectrum sensing is highlighted. In the fourth section, a modification on the AD detector is proposed, taking into account the physical characteristics of spectrum sensing. Finally the performance of the modified detector will be evaluated by simulations.

The Anderson-Darling test: The commonly used test statistic to perform GoF testing was proposed by T. W. Anderson and D. A. Darling in 1952 [5].

$$\frac{1}{n} A_n^2 = \int_0^1 [F_n(x) - F_0(x)]^2 \psi[F_0(x)] dF_0(x), \quad (3)$$

This test statistic is based on the Von Mises criterion, which is an average of the squared discrepancy $[F_n(x) - F_0(x)]^2$, weighted by the increase in $F_0(x)$. The authors in [5] proposed an extra weighting function $\psi(F_0(x))$ to give more importance to the tails of the CDF.

$$\psi[F_0(x)] = \frac{1}{F_0(x)[1 - F_0(x)]} \quad (4)$$

The test statistic given in (3) can be numerically calculate as [6]

$$A_n^2 = -n - \frac{\sum\limits_{i=1}^{n} (2i - 1)(\ln F_0(x_{(i)}) + \ln(1 - F_0(x_{(n+1-i)})))}{n}. \quad (5)$$

In Software Defined Radio technology, the received baseband samples in the digital domain are complex in nature. In this case, the most practical approach to apply the AD test for spectrum sensing is to consider the squared magnitude of the complex samples (i.e. energy of the samples) and test their empirical distribution against the hypothetical noise energy distribution [3]. The sensing method can be summarized as

follows: Consider the classical binary hypothesis test

$$\begin{cases} H_0 : x_i & = & w_i \\ H_1 : x_i & = & r_i + w_i, \end{cases} \quad (6)$$

where r_i are the received complex samples of the transmitted signal, filtered by the channel, and w_i is the complex noise. We now consider the non-negative random variable $Y_i = |x_i|^2$ which corresponds to the received energy and construct the empirical CDF F_n as given in (2). We assume that F_0, representing the hypothetical noise energy distribution, is known or can be estimated. Next, the test statistic A_n^2 is calculated by (5). Once A_n^2 is computed, it will be compared to a predefined threshold λ:

$$\begin{cases} H_0 : A_n^2 \leq \lambda \\ H_1 : A_n^2 > \lambda, \end{cases} \quad (7)$$

The value of λ is determined for a specific value of P_{fa}. A table listing values of λ corresponding to different false alarm probabilities P_{fa} is given in [4]. Otherwise, these values can be computed in advance by Monte Carlo approach. Note that the distribution of A_n^2, and hence the value of λ is independent of the noise distribution represented by F_0.

Problem statement: The problem with the AD test (and also with the Von Mises test) is that the deviation of the empirical CDF $F_n(x)$ to the reference CDF $F_0(x)$ can be either to the left or to the right as the tests are based on the square of the difference $[F_n(x) - F_0(x)]^2$. For most statistical tests, this is convenient, but it is not the case for the application of spectrum sensing.

To illustrate this, let us consider the following simple example. Suppose we want to detect a signal in Gaussian noise. In this case, as the test is performed on the squared values of the received signal, F_0 will be the CDF of the χ_2^2 distribution.

In figure 1 the result of 1000 Monte-Carlo simulations is shown under the H_0 hypothesis (no signal present). For each simulation the binary test (7) is performed with $\lambda = 3.89$, corresponding to a false alarm rate $P_{fa} = 0.01$. The empirical CDFs that are found in the H_0 hypothesis are represented in blue. The reference CDF F_0 for the GoF test corresponding to a χ_2^2 CDF is represented in green. The empirical CDFs that are found in the H_1 hypothesis are in red and represent false alarms. As one can see, in total there are nine false alarms, which corresponds approximately to the pre-setted P_{fa}. Out of the nine, five false alarms are situated to the left of the reference CDF F_0 and four to the right.
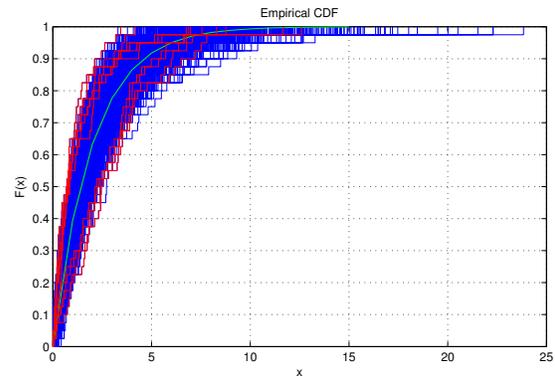
Fig. 1 *Empirical CDFs of Gaussian noise realisations: in blue the empirical CDFs found to be in the H_0 hypothesis, in red the empirical CDFs in the H_1 hypothesis. The reference CDF F_0 is represented in green.*

In case of the proposed AD detector, $Y_i = |x_i|^2$ is a non-negative random variable. From statistics, it is known that the expected value of a non-negative random variable can be written as $E[Y] = \int_0^\infty (1 - F_Y(x)) dx$. As the received signal $\{x_i\}$ has zero means, $E[Y] = E[|x_i|^2] = \sigma_x^2$ also represents the received signal power. Hence, we find

$$\sigma_x^2 = \int_0^\infty (1 - F_Y(x)) dx. \quad (8)$$

In other words, the received signal power equals the area of the region lying above the CDF $F_Y(x)$ and below the line at height 1 to the right of the origin. Based on the same reasoning, we can state that the noise power σ_w^2 equals the area above F_0. This can be seen by replacing in (8) the signal x_i by w_i and F_Y by the expected noise CDF F_0

Coming back to figure 1, five empirical CDFs to the left of F_0 are found to be in the H_1 hypothesis by the AD detector. However, the area above these five empirical CDFs is inferior to the area above F_0. This means that for these empirical CDFs, the power in the received samples x_i is less than the noise power. In the H_1 hypothesis, according to the Bienayme's formula, the power of the received signal $\sigma_x^2 = \sigma_r^2 + \sigma_w^2$, is always larger than the noise power, as signal and noise are uncorrelated. In other words, the five empirical CDFs to the left of F_0 that are found to be in the H_1 hypothesis can be excluded as false alarms. Tagging an empirical CDF left of F_o as a possible signal has no physical meaning as it indicates that, according to (8), the power in the received samples would be less than the expected noise power. As signal and noise are uncorrelated, this is impossible.

Enhancement on the AD test statistic for Spectrum sensing: In order to take into account the observation stated in previous section, we modify the binary test given in (7) by adding a second condition to the H_1 hypothesis

$$\begin{cases} H_1 : A_n^2 > \lambda \text{ and } \sigma_x^2 > \sigma_w^2, \\ H_0 : \text{otherwise} \end{cases} \qquad (9)$$

In [5] it is shown that under the H_0 hypothesis, the random variable $[F_n(x) - F_o(x)]$ approaches a k-variate normal distribution as $n \to \infty$ with $E(F_n(x) - F_o(x)) = 0$. This means that in the H_0 hypothesis, for a given value of x, statistically half of the empirical CDFs $F_n(x)$ will pass above $F_o(x)$ and half will pass underneath. Hence, statistically half of the false alarms are generated by empirical CDFs representing samples x_i with a signal power σ_x^2 less than the expected noise power (i.e. CDFs mainly left of or above F_0) and half by empirical CDFs for which σ_x^2 is larger than the expected noise power (i.e. CDFs mainly to the right of F_0). By excluding in the modified binary test (9) the empirical CDFs which are to the left of F_0, i.e. empirical CDFs for which $\sigma_x^2 > \sigma_w^2$ from the H_1 hypothesis, the false alarm rate will be reduced by a factor of 2.

Simulation: To visualise this gain by a factor of 2 in P_{fa}, we compare in figure 2 the detection performance of both the classical and the modified AD detector as a function of SNR in the range $-20dB$ to $5dB$. For the simulation, λ is set to 2.495 for both detectors. The number of received samples n to calculate the test statistics equals 40. For very small values of SNR, the probability of detection P_d will yield the value of P_{fa}. For the AD detector, according to [4], this value of λ corresponds to a $P_{fa} = 0.05$, which can be verified on figure 2. For the modified AD detector P_d tends to 0.025 for low SNR values, which is half of the pre-setted value of P_{fa}. We also observe that for higher values of SNR, the detection probability of both detectors are equal.

In figure 3 the receiver operating characteristic (ROC) curves of both detectors are represented for an SNR value of $-3dB$ and $n = 40$. Also in these ROC-curves, the gain by the factor of 2 in P_{fa} is very clear. For a given value of P_d, the modified AD detector, based on binary test (9) yields a P_{fa} which is only half the P_{fa} of the classic AD detector based on binary test (7). Vice-versa, for a given value of P_{fa}, the detection probability of the modified AD detector outperforms the detection probability of the AD detector. In summary we can state that for a given value of P_{fa}, the proposed detector is more sensitive than the classical AD detector.

Conclusion: In this paper we studied the Anderson-Darling test as a method for blind spectrum sensing. The AD detector will test the Goodness-of-Fit of empirical distribution of the squared magnitude of the received samples against the noise energy distribution, based on the AD test statistic A_n^2. In this test, the discrepancy between the empirical CDF and the reference CDF can be either to the left or to the right. This means that in the H_0 hypothesis, empirical CDFs that are situated too much left of the reference CDF will also generate false alarms. This has no physical meaning in the application of spectrum sensing as this situation would corresponds to a signal with less power than the expected noise power. In the H_1 hypothesis, the overall power in the received signal should always be larger than the noise power, as noise and signal are uncorrelated. We therefore proposed a modified binary test to eliminate
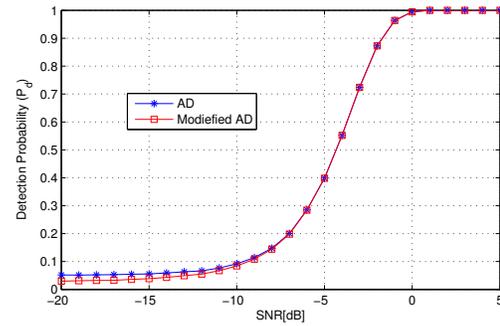
Fig. 2 *Detection probability versus SNR for AD and Enhanced AD detector, $\lambda = 2.495$ for both detectors, $n = 40$.*
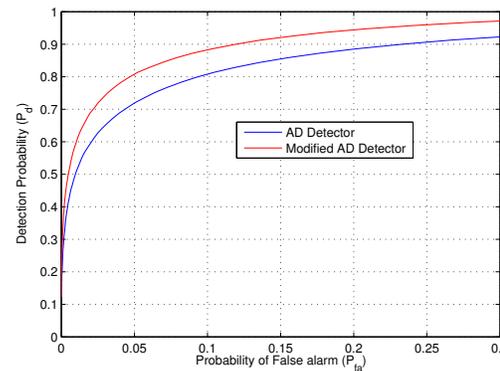
Fig. 3 *Detection probability versus probability of false alarms for AD and Enhanced AD detector, $SNR = -3dB$, $n = 40$.*

the empirical CDFs left to the reference CDF from the H_1 hypothesis, as we know that these empirical CDFs would correspond to a false alarm. Simulations have indeed shown that the modified AD detector reduces the false alarm rate by a factor 2 for a given threshold λ or for a given probability of detection. As a consequence, for a given false alarm rate, the enhanced AD detector is more sensitive than the classic AD detector.

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